Lemaître model and cosmic mass

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Abstract

The mass of a sphere of simmetry in the Lemaître universe is discussed using the Hawking-Hayward quasi-local energy and clarifying existing ambiguities. A covariantly conserved current introduced by Cahill and McVittie is shown to be a multiple of the Kodama energy current.

Keywords: inhomogeneous cosmologies; cosmic mass; cosmic parameters.

1 Introduction

The Lemaître model [1] is a spherically symmetric inhomogeneous universe which solves the Einstein equations and generalizes the better known Lemaître-Tolman-Bondi (LTB) [2, 3] geometry to the case of non-zero pressure. Inhomogeneous universes (see Ref. [4] for a comprehensive review) have been the subject of much work in recent years because of the attempts to explain the current acceleration of the universe without an *ad hoc* dark energy or cosmological constant and without abandoning general relativity (see [5] for a review). This work on LTB models has revived interest also in Lemaître models and inhomogeneous cosmologies in general. The issue has been raised recently of what is the physical mass contained in a sphere in the Lemaître universe [6]. Over the years, there have been various proposals (reviewed in [6]). More recently, LTB models have been studied in various contexts [7] and the study has moved to a higher degree of sophistication with the introduction of perturbations of LTB models [8]; similar to the case of Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, perturbations allow research on more sophisticated physics and open the door to the use of tools such as the temperature fluctuations in the cosmic microwave background [9]. As a starting point for this and future developments, it would be reassuring to know that we understand the physics of the *unperturbed* Lemaître universe, including its energy. Here we show that approaching this issue using the Hawking-Hayward quasi-local energy [10, 11] and the Kodama vector [12] dissipates some ambiguities in the recent literature. In the presence of spherical symmetry (and, therefore, also in the Lemaître universe), the Hawking-Hayward quasi-local energy reduces [11, 17] to the Misner-Sharp-Hernandez mass [13]. Spherical symmetry allows one to introduce also the Kodama vector [12], which generates a covariantly conserved current and has the Misner-Sharp-Hernandez mass as its Noether charge [11].

In the next section we discuss the Lemaître universe and the Misner-Sharp-Hernandez mass contained in sphere of symmetry,[1] showing how this clarifies doubts that still linger on in the literature. The following section discusses the Kodama vector and shows that a conserved current discussed in Refs. [14, 6] is a multiple of the Kodama current built out of the Einstein tensor and the Kodama vector.

We use units in which the speed of light and Newton's constant are unity and we follow the notation of Wald's textbook [15].

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[1]By "sphere of symmetry" we mean a 2-dimensional surface which is an orbit of the isometry of the spacetime manifold describing spherical symmetry (of course, such orbits exist through any point of a spherically symmetric spacetime).

2 Lemaître geometry and the mass of a sphere of symmetry

The spherically symmetric Lemaître line element in coordinates $\{x^\mu\} = \{t, r, \theta, \varphi\}$ comoving with the fluid source is [1]

$$ds^2 = -e^{2\sigma}dt^2 + e^\lambda dr^2 + R^2 d\Omega^2_{(2)}, \tag{2.1}$$

where $\sigma = \sigma(t, r)$, $\lambda = \lambda(t, r)$, and $R(t, r)$ is the areal radius, while $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the line element on the unit 2-sphere. The simplest stress-energy tensor sourcing the Lemaître spacetime is that of a perfect fluid

$$T_{ab} = (P + \rho)\, u_a u_b + P g_{ab} \tag{2.2}$$

where $\rho(t, r)$ and $P(t, r)$ are the energy density and pressure of the fluid as perceived by a comoving observer with 4-velocity u^c with components $u^\mu = (e^{-\sigma}, 0, 0, 0)$ in comoving coordinates. In general, however, the matter source of the Lemaître metric (2.1) is not restricted to be a perfect fluid.

Following [6], we assume isotropic pressure (which is a restriction on the full Lemaître spacetime) and we allow for a cosmological constant Λ. The Lemaître universe is a solution of the Einstein equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \tag{2.3}$$

where G_{ab} is the Einstein tensor and g_{ab} is the spacetime metric.

In order to discuss the Misner-Sharp-Hernandez mass M_{MSH} and the Kodama vector k^a, it is convenient to express the line element (2.1) using the areal radius $R = x^{1'}$ instead of $r = x^1$ because then M_{MSH} and k^a assume straightforward expressions. Our conclusions, however, are fully coordinate-independent and it will be easy to revert to the coordinates $\{x^\mu\} = \{t, r, \theta, \varphi\}$. Our goal is to recast the line element (2.1) in the form

$$ds^2 = -A(T, R)dT^2 + B(T, R)dR^2 + R^2 d\Omega^2_{(2)}. \tag{2.4}$$

Using the differential relation $dR = \dot{R}dt + R'dr$, where an overdot and a prime denote differentiation with respect to t and r, respectively, the line element (2.1) becomes

$$ds^2 = -\left(e^{2\sigma} - e^\lambda \frac{\dot{R}^2}{R'^2}\right) dt^2 + \frac{e^\lambda}{R'^2} dR^2 - \frac{2\dot{R}\, e^\lambda}{R'^2} dt dR + R^2 d\Omega^2_{(2)}. \tag{2.5}$$

The cross-term in $dtdR$ is eliminated by introducing a new time coordinate $x^{0'} = T$ defined by

$$dT = \frac{1}{F}\left(dt + \beta dR\right), \tag{2.6}$$

where $\beta(t, r)$ is a function to be determined and $F(t, r)$ is an integrating factor which must be introduced to guarantee that dT is an exact differential. It satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{F}\right) = \frac{\partial}{\partial t}\left(\frac{\beta}{F}\right). \tag{2.7}$$

Upon substitution of $dt = FdT - \beta dR$, the line element (2.5) assumes the form

$$
\begin{aligned}
ds^2 &= -\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)F^2dT^2 + 2F\left[\beta\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right) - \frac{\dot{R}}{R'^2}e^{\lambda}\right]dTdR \\
&\quad + \left[\frac{e^{\lambda}}{R'^2} - \left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)\beta^2 + 2\beta\frac{\dot{R}}{R'^2}e^{\lambda}\right]dR^2 + R^2d\Omega^2_{(2)}.
\end{aligned} \tag{2.8}
$$

Setting

$$\beta(t, r) = \frac{\dot{R}e^{\lambda}}{R'^2\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)} \tag{2.9}$$

one obtains

$$ds^2 = -\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)F^2dT^2 + \frac{e^{\lambda}e^{2\sigma}}{R'^2\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)}dR^2 + R^2d\Omega^2_{(2)}, \tag{2.10}$$

which is of the form (2.4) with

$$A(T, R) = \left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)F^2, \tag{2.11}$$

$$B(T, R) = \frac{e^{\lambda}e^{2\sigma}}{R'^2\left(e^{2\sigma} - e^{\lambda}\frac{\dot{R}^2}{R'^2}\right)}. \tag{2.12}$$

In his original paper, Lemaître identified (without a real justification) the mass contained in a sphere of radius r with the quantity

$$M = \frac{R}{2}\left(1 + \dot{R}^2e^{-2\sigma} - R'^2e^{-\lambda} - \frac{\Lambda R^2}{3}\right). \tag{2.13}$$

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The derivatives of this "mass" M are related to the cosmic fluid density and pressure by [1, 6]

$$M' = 4\pi R^2 R' \rho \,, \tag{2.14}$$

$$\dot{M} = -4\pi R^2 \dot{R} P \,. \tag{2.15}$$

In 1970, Cahill and McVittie [14] studied the Lemaître model with $\Lambda = 0$ and identified the right hand side of eq. (2.13) (with $\Lambda = 0$) with the physical mass. The rationale was that, if a Lemaître model is joined to an exterior Schwarschild geometry in a Swiss-cheese model, then M must equal the exterior mass,[2] and that the Bianchi identities generalize eqs. (2.14) and (2.15). Cahill and McVittie were aware that their mass proposal M coincided with the Misner-Sharp-Hernandez mass, then just recently introduced [14].

The authors of Ref. [6] note that the interpretation of M is rather clear when there is no pressure and Λ is absent, but it is not so straighforward otherwise. They proceed to note that eq. (2.14), which does not contain P, holds in all Lemaître models. This is a good observation but, in principle, there could be other quantities or equations which do not depend explicitly on P and could be used to define effective masses. As a guideline to find out the physical mass of sphere, the authors of [6] proceed to analyze the geodesic deviation equation in order to establish which mass is "seen" by test particles, and the discussion necessarily becomes rather involved. They reach the rather disheartening conclusion that "we cannot separate the mass, the cosmological constant, the density and the pressure from each other, and so we cannot create a unique definition of mass based on geometric invariants of the metric in the general case" [6]. Indeed, we can. Our knowledge of energy and mass in general relativity has progressed greatly since the times of Lemaître and Cahill and McVittie, with the introduction of the various quasi-local energies (see [16] for a review), which culminated in the Hawking-Hayward quasi-local energy [10, 11]. In spherical symmetry, the Hawking-Hayward quasi-local energy reduces [11, 17] to the Misner-Sharp-Hernandez mass M_{MSH} [13], which is defined in a coordinate-invariant way by [13, 11, 17, 21, 22, 27]

$$1 - \frac{2M_{\mathrm{MSH}}}{R} = \nabla^c R \nabla_c R \,, \tag{2.16}$$

where R is the areal radius (which, being related to the area $\mathcal{A}$ of 2-spheres of symmetry by $R = \sqrt{\frac{\mathcal{A}}{4\pi}}$, is a geometrically defined quantity).

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[2]In retrospect, this is a good argument because there is little arguing on the physical mass of the Schwarzschild spacetime, and the choice proved to give the correct answer (see below).

In coordinates $\{x^{\mu'}\} = \{T, R, \theta, \varphi\}$, the squared gradient in the right hand side of eq. (2.16) is simply $g^{RR} = B^{-1}$ and gives the Misner-Sharp-Hernandez mass

$$
\begin{aligned}
M_{\text{MSH}} &= \frac{R}{2}\left(1 - g^{RR}\right) \\
&= \frac{R}{2}\left(1 + \dot{R}^2 \text{e}^{-2\sigma} - R'^2 \,\text{e}^{-\lambda}\right) \\
&= M + \frac{\Lambda R^3}{6}. \quad (2.17)
\end{aligned}
$$

So, for $\Lambda = 0$, the Cahill-McVittie prescription coincides with the Hawking-Hayward/Misner-Sharp-Hernandez one. This result can, of course, be obtained also in coordinates $\{t, r, \theta, \varphi\}$ by computing $\nabla^c R \nabla_c R$ with the metric (2.1) and using the well known relation

$$
\dot{R} = \pm \text{e}^{\sigma} \sqrt{\frac{2M}{R} + R'^2 \,\text{e}^{-\lambda} - 1 + \frac{\Lambda R^2}{3}}, \quad (2.18)
$$

which follows from the definition (2.13) [6].

Note that the Misner-Sharp-Hernandez mass does not depend explicitly on the pressure P (although P affects the cosmic expansion and determines the metric coefficients which, in turn, determine M_{MSH}), while $\dot{M}$ depends on P but not on ρ (eq. (2.15)). This fact is well known in FLRW space [11, 18], to which the Lemaître model reduces if $\sigma \equiv 1$ and $\lambda = \lambda(t)$. Therefore, there is no issue of disentangling the contribution of P from those of ρ and Λ. The contribution $\Lambda R^3/6$ to M_{MSH} is easily interpreted as the mass corresponding to the cosmological constant energy density $\rho_\Lambda = \frac{\Lambda}{8\pi}$ contained in a sphere of areal radius R and volume $4\pi R^3/3$. The decomposition of M_{MSH} into a "local" and a "cosmological" part is covariant: this point has been discussed in detail in [19] for the McVittie metric [20], which exhibits some of the properties of the Lemaître model (although it belongs to a different family of solutions of the Einstein equations), and we will not repeat the discussion here.

Although the Hawking-Hayward mass is not mentioned in [6], the authors somehow end up reasoning along the same lines in their quest for the physical mass, when they stress the role of the apparent horizon in relating cosmic mass and diameter distance. In fact, the apparent horizons of any spherically symmetric metric are defined by $\nabla^c R \nabla_c R = 0$ (*e.g.*, [21, 22]), which relates the apparent horizon radii with the Misner-Sharp-Hernandez mass contained through the relation $R_{\text{AH}} = 2M_{\text{MSH}}$ (which mimics the expression of the Schwarzschild radius in the Schwarzschild spacetime) [21, 22]. In the

Lemaître model, the recipe to locate the apparent horizons translates into $g^{RR} = 0$, or

$$R'^2 \mathrm{e}^{2\sigma} - \dot{R}^2 \mathrm{e}^{\lambda} = 0 \,. \tag{2.19}$$

In general, multiple solutions to this equation (which must be solved numerically) are possible, describing both black hole and cosmological apparent horizons which evolve in time (examples are solved in [23, 24]).

Using the Misner-Sharp-Hernandez mass (2.17), the line element (2.10) can now be written as

$$ds^2 = -\mathrm{e}^{2\sigma+\lambda} \left(\frac{F}{R'}\right)^2 \left(1 - \frac{2M_{\mathrm{MSH}}}{R}\right) dT^2 + \frac{dR^2}{1 - \frac{2M_{\mathrm{MSH}}}{R}} + R^2 d\Omega_{(2)}^2 \,. \tag{2.20}$$

The spatial part of this line element resembles the spatial part of the Schwarzschild metric, but only superficially because M_{MSH} is not a constant but depends on the areal radius R.

3   The Kodama energy current

In a generic Lemaître model there is no timelike Killing vector but, as in any spherically symmetric spacetime, one can introduce the closest thing to it, the Kodama vector [12]

$$k^a = \epsilon^{ab} \nabla_b R \,, \tag{3.1}$$

where ϵ_{ab} is the volume element on the 2-space orthogonal to the 2-spheres of symmetry. If the spacetime metric is decomposed according to

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega_{(2)}^2 \qquad (a, b = 0, 1) \,, \tag{3.2}$$

then ϵ_{ab} is the volume form associated with the 2-metric h_{ab}. The Kodama vector plays the role of a Killing vector where there is none: it is timelike in a spacetime region, null on an apparent horizon, and becomes spacelike on the other side of this horizon [12]. What makes the Kodama vector remarkable is the fact that the Kodama current

$$J^a = G^{ab} k_b \tag{3.3}$$

associated with it is covariantly conserved [12], $\nabla_c J^c = 0$, such an unexpected property to be called the "Kodama miracle" [22]. What is more, the Misner-Sharp-Hernandez mass almost universally identified with the physical mass-energy in spherical symmetry in general relativity, turns out to be the Noether charge associated with the Kodama

current [11]. There are strong claims in the literature that, in the absence of a preferred time derived from a timelike Killing vector, the Kodama vector introduces a preferred time and surface gravity on apparent horizons, which determine a Hawking temperature and make thermodynamics meaningful for time-evolving *apparent* horizons (see Ref. [25, 26, 27] for reviews). Given the geometry (2.4), the Kodama vector is [12]

$$k^a = \frac{-1}{\sqrt{AB}} \left(\frac{\partial}{\partial T} \right)^a \tag{3.4}$$

and, in the Lemaître geometry (2.10), it has components

$$k^{\mu'} = -\frac{|R'| e^{-\sigma} e^{-\lambda/2}}{F} \delta^{\mu'0} . \tag{3.5}$$

Cahill and McVittie found a conserved 4-current (reported also in [6]) which, in comoving coordinates $\{x^\mu\} = \{t, r, \theta, \varphi\}$, has components

$$J^\mu_{\text{(CM)}} = \frac{\sin\theta}{4\pi\sqrt{-g}} \left(M', -\dot{M}, 0, 0 \right) = \frac{e^{-\sigma} e^{-\lambda/2}}{4\pi R^2} \left(M', -\dot{M}, 0, 0 \right) . \tag{3.6}$$

Eqs. (2.14) and (2.15) then yield

$$J^\mu_{\text{(CM)}} = e^{-\sigma} e^{-\lambda/2} \left(R'\rho, \dot{R}P, 0, 0 \right) . \tag{3.7}$$

It is natural to ask whether this current is the same as the Kodama current. To find out, one computes the Kodama vector (3.1), which is found to have components in comoving coordinates

$$k^\mu = \epsilon^{\mu\nu} \nabla_\nu R = \epsilon^{\mu0} \dot{R} + \epsilon^{\mu1} R' . \tag{3.8}$$

Therefore, it is

$$k^0 = \epsilon^{01} R' = g^{00} g^{11} \sqrt{|h|} R' = -e^{-\sigma} e^{-\lambda/2} R' , \tag{3.9}$$

$$k^1 = \epsilon^{10} \dot{R} = -g^{00} g^{11} \sqrt{|h|} \dot{R} = e^{-\sigma} e^{-\lambda/2} \dot{R} , \tag{3.10}$$

where $h = -e^{2\sigma} e^\lambda$ is the determinant of the 2-metric h_{ab} in the submanifold orthogonal to the 2-spheres of symmetry. We have

$$k^\mu = e^{-\sigma} e^{-\lambda/2} \left(-R', \dot{R}, 0, 0 \right) \tag{3.11}$$

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and, lowering the indices,

$$k_\mu = \left(e^\sigma e^{-\lambda/2} R', e^{-\sigma} e^{\lambda/2} \dot{R}, 0, 0 \right) . \tag{3.12}$$

Using the Einstein equations (2.3) and the fluid four-velocity $u_\mu = -e^\sigma \delta_{\mu 0}$, the non-vanishing components of the Kodama current $J^a = G^{ab} k_b$ are found to be

$$J^0 = G^{00} k_0 = (g^{00})^2 G_{00} k_0 = e^{-\sigma} e^{-\lambda/2} R' (8\pi\rho + \Lambda) , \tag{3.13}$$

$$J^1 = G^{11} k_1 = (g^{11})^2 G_{11} k_1 = e^{-\sigma} e^{-\lambda/2} \dot{R} (8\pi P - \Lambda) , \tag{3.14}$$

By comparing the expression

$$J^\mu = e^{-\sigma} e^{-\lambda/2} \left(R' (8\pi\rho + \Lambda) , \dot{R} (8\pi P - \Lambda) , 0, 0 \right) \tag{3.15}$$

with eq. (3.7), it is clear that the Cahill-McVittie conserved current is just

$$J^\mu_{(\text{CM})} = (8\pi)^{-1} J^\mu \tag{3.16}$$

when $\Lambda = 0$ (and it is obtained by contracting T^{ab}, instead of G^{ab}, with the Kodama vector). This fact was to be suspected, since asking for two separate "miracles" would be asking too much.

4   Conclusions

There are several quasi-local energy constructs in general relativity (see the review [16]) and there is no mathematical "proof" selecting the "correct" one. However, there is now general consensus that the Hawking-Hayward quasi-local construction [10, 11] is preferred in the sense that it encapsulates better than its competitors the physical properties required by the mass-energy of a system. Among the advantages of the Hawking-Hayward mass notion are the facts that it is well defined for non-asymptotically flat and non-stationary spacetimes [16]. In spherical symmetry, the Hawking-Hayward quasi-local energy reduces [17] to the better known Misner-Sharp-Hernandez mass [13]. It appears that Cahill and McVittie identified the correct mass notion in the $\Lambda = 0$ Lemaître space, the Hawking-Hayward/Misner-Sharp-Hernandez one. It is well known since the early days of the Hawking-Hayward mass [11, 17] that this object is also the Noether charge associated with the Kodama current, and we won't repeat the derivation of this result here. The Cahill-McVittie covariantly conserved current is just a multiple

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of the Kodama energy current. Lemaître, Cahill, and McVittie made a clever choice and they were correct after all, although perhaps they could not be sure of the reason why, because they came decades before Hawking and Hayward or before the present-day understanding of the quasi-local energy.

Acknowledgments

References

[1] Lemaître, G., *Ann. Soc. Sci. Bruxelles A* **53**, 51 (1933) [reprinted in *Gen. Rel. Grav.* **29**, 641 (1997)]

[2] Tolman, R.C., *Proc. Nat. Acad. Sci U.S.A.* **20**, 169 (1934)

[3] Bondi, H., *Mon. Not. R. Astron. Soc.* **107**, 410 (1947)

[4] Krasiński, A., *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997)

[5] Bolejko, K., Celerier, M.-N., and Krasiński, A., *Class. Quantum Grav.* **28**, 164002 (2011)

[6] Alfedeel, A.A.H. and Hellaby, C., *Gen. Rel. Grav.* **42**, 1935 (2010)

[7] Pavlidou, V., Tetradis, N., and Tomaras, T.N., *JCAP* **1405**, 017 (2014); Bolejko, K., Hellaby, C., and Alfedeel, A.H.A., *JCAP* **1109**, 011 (2011); Nogueira, F.A.M.G., arXiv:1312.5005; Iribarrem, A. *et al.*, arXiv:1308.2199; Marra, V. and Paakkonen, M. *JCAP* **1201**, 025 (2012); Clarkson, C. and Regis, M., *JCAP* **1102**, 013 (2011); Clarkson, C. and Maartens, R., *Class. Quantum Grav.* **27**, 124008 (2010); Moradi, R., Firouzjaee, J.T., and Mansouri, R., arXiv:1301.1480

[8] Leithes, A. and Malik, K.A., arXiv:1403.7661

[9] Durrer, R., *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, 2008)

[10] Hawking, S.W., *J. Math. Phys.* **9**, 598 (1968)

[11] Hayward, S.A., *Phys. Rev. D* **49**, 831 (1994)

[12] Kodama, H., *Prog. Theor. Phys.* **63**, 1217 (1980)

[13] Misner, C.W. and Sharp, D.H., *Phys. Rev.* **136**, B571 (1964); Hernandez, W.C. and Misner, C.W., *Astrophys. J.* **143**, 452 (1966)

[14] Cahill, M.E. and McVittie, G.C., *J. Math. Phys.* **11**, 1382 (1970)

[15] Wald, R.M., *General Relativity* (University of Chicago Press, Chicago, 1984)

[16] Szabados, L. *Living Rev. Relat.* **7**, 4 (2004)

[17] Hayward, S.A., *Phys. Rev. D* **53**, 1938 (1996)

[18] Faraoni, V. *Phys. Rev. D* **84**, 024003 (2011)

[19] Carrera, M. and Giulini, D., *Rev. Mod. Phys.* **82**, 169 (2010)

[20] McVittie, G.C., *Mon. Not. R. Astron. Soc.* **93**, 325 (1933)

[21] Nielsen A.B. and Visser, *Class. Quantum Grav.* **23**, 4637 (2006)

[22] Abreu, G. and Visser, M., *Phys. Rev. D* **82**, 044027 (2010)

[23] Booth, I., Brits, L, Gonzalez, J.A., and van den Broeck, V., *Class. Quantum Grav.* **23**, 413 (2006)

[24] Gao, C., Chen, X., Shen, Y.-G., and Faraoni, V., *Phys. Rev. D* **84**, 104047 (2011)

[25] Di Criscienzo, R., Hayward, S.A., Nadalini, M., Vanzo, L., and Zerbini, S., *Class. Quantum Grav.* **27**, 015006 (2010)

[26] Faraoni, V., *Galaxies* **1**, 114 (2013) [arXiv:1309.4915]

[27] Faraoni, V. *Cosmological and Black Hole Apparent Horizons* (Springer, New York, forthcoming 2015)